

Note on Exchange Gate on the Qudit Space

Kazuyuki FUJII *

Department of Mathematical Sciences

Yokohama City University

Yokohama, 236-0027

Japan

Abstract

We construct the exchange gate with small elementary gates on the space of qudits, which consist of three controlled shift gates and three "reverse" gates. This is a natural extension of the qubit case.

We also consider a similar subject on the Fock space, but in this case it is not clear what elementary gates are. We present some related problems.

1 Introduction

The purpose of this note is to construct the exchange gate with small elementary gates on the space of qudits. This is a "natural" extension in the qubit case.

The standard model of Quantum Computation is based on 2-level quantum systems (qubits) which are natural counterpart to the classical systems. In it many quantum logic gates such as controlled NOT ones, exchange one have been constructed, see [1], [4] or [5], [7]. The universality of quantum computation has been proved.

We have a beautiful theory, [15].

A major obstacle to standard quantum computation is the problem of decoherence depending deeply on increasing of coupled qubits. This problem is inevitable to every quantum system and is not easy to overcome, [20].

To reduce the number of coupled qubits the non-standard model of quantum computation based on d -level quantum systems (qudits) has been considered, see for example [3], [2] and [14]. This theory might become useful in the near future.

However it seems (at least) to the author that there are some defects in this theory. Let us explain. The controlled shift gates corresponding to controlled NOT ones have been defined, but general controlled unitary gates have not been constructed. Moreover the universality on 2-qudit space has not been proved as far as the author knows.

The exchange gate plays an important role in Quantum Computation. In this note we focus our attention to this topics. It is in the qubit case constructed by making use

*E-mail address : fujii@yokohama-cu.ac.jp

of three controlled NOT gates. This is a well-known result. We would like to consider a similar problem in the qudit case. In the case we can construct the exchange gate by making use of three controlled shift gates and three "reverse" gates. This is our main result.

We would like to extend our problem to the case of Fock space. Namely, we want to construct the exchange operator with some elementary operators (corresponding to elementary gates in the finite-dimensional case). However we meet some trouble. We cannot define controlled shift operators as unitary ones due to the infinite-dimensional property. What elementary gates (operators) is in the infinite-dimensional case is not clear.

Instead of taking all quantum states into consideration let us restrict our target to small set of (important) quantum ones, for example, coherent states or squeezed-coherent states which are famous in Quantum Optics, [16], [19] or [6]. We can construct the operator exchanging coherent states by making use of generalized coherent operator based on Lie algebra $su(2)$ and some phase operators. But at this stage we cannot do the operator exchanging squeezed-coherent states. This is dissatisfied to us. We need further study.

Here let us state our hiding purpose. We are developing a geometric method in quantum computation called Holonomic Quantum Computation, [8], [9], [10] and [21], [17]. This is of course based on the qubit space, so we would like to generalize it on the qudit space. In the forthcoming papers we will report this task.

2 Exchange Gate

In this section we make a brief review of the construction of exchange gate on the qubit space and next extend the method from there to on the qudit space.

2.1 Qubit Case · · · Review

The 1-qubit space is identified with \mathbf{C}^2 with basis $\{|0\rangle, |1\rangle\}$;

$$\mathbf{C}^2 = \text{Vect}_{\mathbf{C}}\{|0\rangle, |1\rangle\}, \quad |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We consider the abelian group \mathbf{Z}_2 with group operation \oplus

$$a \oplus b = a + b \pmod{2}. \tag{1}$$

Explicitly

$$0 \oplus 0 = 0, \quad 0 \oplus 1 = 1, \quad 1 \oplus 0 = 1, \quad 1 \oplus 1 = 0.$$

From this it is easy to see

$$a \oplus b = a + b - 2ab = a + (-1)^{ab}b.$$

The t -qubit space is the **tensor product** (not direct sum) of \mathbf{C}^2

$$\mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \cdots \otimes \mathbf{C}^2 \equiv (\mathbf{C}^2)^{\otimes t} \tag{2}$$

with basis

$$\{|n_1, n_2, \dots, n_t\rangle = |n_1\rangle \otimes |n_2\rangle \otimes \dots \otimes |n_t\rangle \mid n_j \in \mathbf{Z}_2 = \{0, 1\}\}.$$

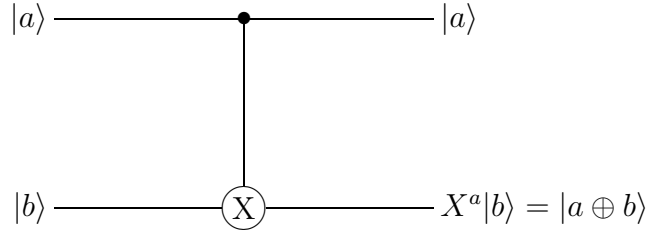
For example,

$$|0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |0\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |1\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Now we consider a controlled NOT operation (gate) which we will denote by C_X in the following. It is defined by

$$C_X : |a\rangle \otimes |b\rangle \longrightarrow |a\rangle \otimes X^a|b\rangle = |a\rangle \otimes |a \oplus b\rangle, \quad a, b \in \mathbf{Z}_2. \quad (3)$$

Graphically it is expressed as



and the matrix representation is

$$C_X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (4)$$

Here we note the relation

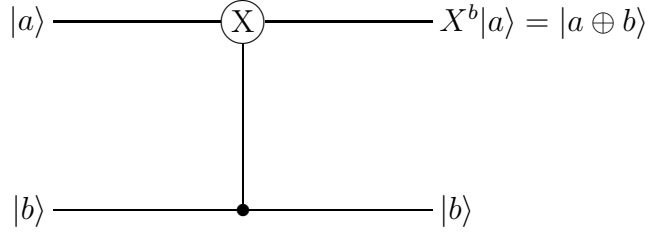
$$X^a|b\rangle \equiv \sigma_1^a|b\rangle = |a \oplus b\rangle \quad \text{for } a, b \in \mathbf{Z}_2. \quad (5)$$

where σ_1 is the Pauli matrix.

In this case the first bit is called a control bit and the second a target bit. Of course we can consider the reverse case. Namely, the first bit is a target one and the second a control one, which is also called the controlled NOT operation :

$$\tilde{C}_X : |a\rangle \otimes |b\rangle \longrightarrow X^b|a\rangle \otimes |b\rangle = |a \oplus b\rangle \otimes |b\rangle, \quad a, b \in \mathbf{Z}_2, \quad (6)$$

and graphically it is expressed as



and the matrix representation is

$$\tilde{C}_X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \quad (7)$$

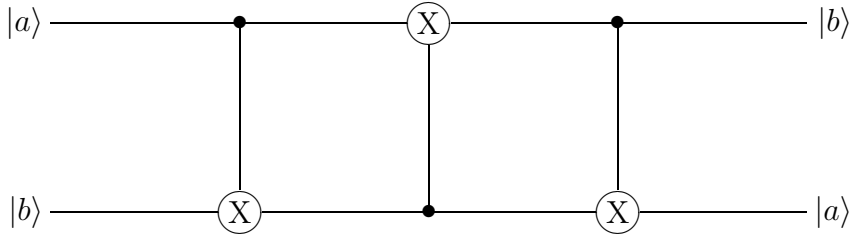
The exchange (swap) gate is defined by

$$S(|a\rangle \otimes |b\rangle) = |b\rangle \otimes |a\rangle \quad \text{for } a, b \in \mathbf{Z}_2. \quad (8)$$

It is well-known that this gate is constructed by making use of three controlled NOT gates

$$S = C_X \circ \tilde{C}_X \circ C_X \quad (9)$$

and is graphically expressed as



2.2 Qudit Case

The 1-qudit space is identified with \mathbf{C}^d with basis $\{|0\rangle, |1\rangle, \dots, |d-1\rangle\}$;

$$\mathbf{C}^d = \text{Vect}_{\mathbf{C}}\{|0\rangle, |1\rangle, \dots, |d-1\rangle\}, \quad |j\rangle = (0, \dots, 0, 1, 0, \dots, 0)^T \quad (j - \text{position}),$$

where T is the transpose of a vector.

We consider the abelian group \mathbf{Z}_d with group operation \oplus

$$a \oplus b = a + b \pmod{d}. \quad (10)$$

Here we deal with two elementary operations $\{\Sigma_1, \Sigma_3\}$ in the unitary group $U(d)$ which correspond to the Pauli matrices $\{\sigma_1, \sigma_3\}$. They are defined as

$$\Sigma_1|a\rangle = |a \oplus 1\rangle, \quad \Sigma_3|a\rangle = \zeta^a|a\rangle \quad \text{for } a \in \mathbf{Z}_d, \quad (11)$$

where $\zeta = \exp(2\pi i/d)$. We note that they are not hermitian

$$\Sigma_1^\dagger = \Sigma_1^{d-1}, \quad \Sigma_3^\dagger = \Sigma_3^{d-1}, \quad \text{and} \quad \Sigma_3 \Sigma_1 = \zeta \Sigma_1 \Sigma_3.$$

In the following we set $\Sigma = \Sigma_1$ for simplicity.

The controlled shift gates are defined as

$$C_\Sigma(|a\rangle \otimes |b\rangle) = |a\rangle \otimes \Sigma^a |b\rangle = |a\rangle \otimes |a \oplus b\rangle \quad (12)$$

and

$$\tilde{C}_\Sigma(|a\rangle \otimes |b\rangle) = \Sigma^b |a\rangle \otimes |b\rangle = |a \oplus b\rangle \otimes |b\rangle. \quad (13)$$

The exchange (swap) gate which is our target in this paper is given by

$$S = \sum_{a,b=0}^{d-1} (|a\rangle \otimes |b\rangle)(\langle b| \otimes \langle a|) = \sum_{a,b=0}^{d-1} |a\rangle \langle b| \otimes |b\rangle \langle a|. \quad (14)$$

Problem How can we construct this gate with some elementary gates like (9) ?

Before dealing with this problem let us define an important operation K in $U(d)$:

$$K(|a\rangle) = |a \oplus a \oplus \cdots \oplus a \text{ (} d-2 \text{ times)}\rangle \quad \text{for} \quad a \in \mathbf{Z}_d. \quad (15)$$

Since

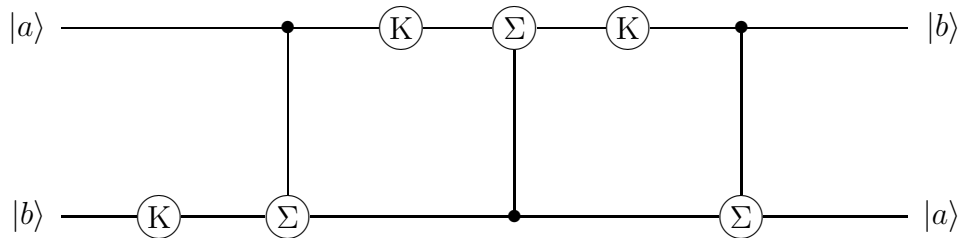
$$a \oplus a \oplus \cdots \oplus a = \begin{cases} 0 & \text{if } a = 0 \\ d - a & \text{if } a \neq 0 \end{cases}$$

, we have $K(|0\rangle) = |0\rangle$ and $K(|a\rangle) = |d - a\rangle$ for $1 \leq a \leq d - 1$. This is a kind of reverse operation. We note that when $d = 2$ this operation is just the identity, so we cannot see it in the figure.

Now we are in a position to state our main result. The exchange (swap) gate is constructed as

$$S = C_\Sigma \circ (K \otimes 1) \circ \tilde{C}_\Sigma \circ (K \otimes 1) \circ C_\Sigma \circ (1 \otimes K) \quad (16)$$

and it is graphically expressed as



Our result is rather complicated compared with (9).

2.3 Problem

Here let us present an important problem related to the universality of 2-qudit spaces. We usually assume that in the quantum computation

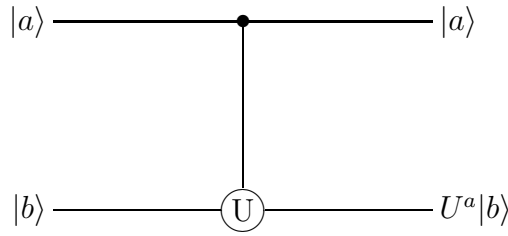
- (1) each qudit space is universal,
- (2) controlled shift gates can be constructed (efficiently).

Here the terminology "universal" means that any unitary element in $U(d)$ can be constructed.

Now we define controlled unitary gates

$$C_U(|a\rangle \otimes |b\rangle) = |a\rangle \otimes U^a|b\rangle, \quad (17)$$

where $U \in U(d)$.



Another controlled unitary gates \tilde{C}_U defined by $\tilde{C}_U(|a\rangle \otimes |b\rangle) = U^b|a\rangle \otimes |b\rangle$ are obtained easily

$$S \circ C_U \circ S = \tilde{C}_U \quad (18)$$

by making use of the exchange gate constructed above.

Problem (A) Under the above assumption (1) and (2) can all controlled unitary gates (operations) be constructed ?

This is a crucial problem. Moreover

Problem (B) Is the 2-qudit space universal ? or Can every unitary operation in $U(d^2)$ be constructed under the assumption ?

As far as we know these problems have not been solved, [11].

3 Exchange Operator in the Fock Space

We make a short review of general theory of both a coherent operator and generalized coherent one based on Lie algebra $su(2)$, and explain our result and present the related problems.

Let $a(a^\dagger)$ be the annihilation (creation) operator of the harmonic oscillator. If we set $N \equiv a^\dagger a$ (: number operator), then

$$[N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \quad [a^\dagger, a] = -\mathbf{1}. \quad (19)$$

Let \mathcal{H} be a Fock space generated by a and a^\dagger , and $\{|n\rangle \mid n \in \mathbf{N} \cup \{0\}\}$ be its basis. The actions of a and a^\dagger on \mathcal{H} are given by

$$a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad N|n\rangle = n|n\rangle \quad (20)$$

where $|0\rangle$ is a normalized vacuum ($a|0\rangle = 0$ and $\langle 0|0\rangle = 1$). From (20) state $|n\rangle$ for $n \geq 1$ are given by

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle. \quad (21)$$

These states satisfy the orthogonality and completeness conditions

$$\langle m|n\rangle = \delta_{mn} \quad \text{and} \quad \sum_{n=0}^{\infty} |n\rangle\langle n| = \mathbf{1}. \quad (22)$$

We call a state defined by

$$|z\rangle = e^{za^\dagger - \bar{z}a}|0\rangle \equiv D(z)|0\rangle \quad \text{for } z \in \mathbf{C} \quad (23)$$

the coherent state.

Next let us define a squeezed operator $S(w)$. If we set

$$K_+ \equiv \frac{1}{2}(a^\dagger)^2, \quad K_- \equiv \frac{1}{2}a^2, \quad K_3 \equiv \frac{1}{2}\left(a^\dagger a + \frac{1}{2}\right), \quad (24)$$

then it is easy to check the $su(1,1)$ relations

$$[K_3, K_+] = K_+, \quad [K_3, K_-] = -K_-, \quad [K_+, K_-] = -2K_3. \quad (25)$$

That is, the set $\{K_+, K_-, K_3\}$ gives a unitary representation of $su(1,1)$ with spin $K = 1/4$ and $3/4$, [18]. We call an operator

$$S(w) = e^{wK_+ - \bar{w}K_-} = e^{\frac{1}{2}\{w(a^\dagger)^2 - \bar{w}a^2\}} \quad \text{for } w \in \mathbf{C} \quad (26)$$

the squeezed operator.

The operators $D(z)$ and $S(w)$ play a crucial role in Quantum Optics, [16] and [19].

Next we consider the system of two-harmonic oscillators. If we set

$$a_1 = a \otimes 1, \quad a_1^\dagger = a^\dagger \otimes 1; \quad a_2 = 1 \otimes a, \quad a_2^\dagger = 1 \otimes a^\dagger, \quad (27)$$

then it is easy to see

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0, \quad [a_i, a_j^\dagger] = \delta_{ij}, \quad i, j = 1, 2. \quad (28)$$

We also denote by $N_j = a_j^\dagger a_j$ ($j = 1, 2$) number operators.

Now we can construct representation of Lie algebras $su(2)$ by making use of Schwinger's boson method, see [12], [13]. Namely if we set

$$J_+ = a_1^\dagger a_2, \quad J_- = a_2^\dagger a_1, \quad J_3 = \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2), \quad (29)$$

then it is easy to check that (29) satisfies the $su(2)$ relations

$$[J_3, J_+] = J_+, \quad [J_3, J_-] = -J_-, \quad [J_+, J_-] = 2J_3. \quad (30)$$

In the following we define (unitary) generalized coherent operators based on Lie algebras $su(2)$.

We set

$$U_J(z) = e^{za_1^\dagger a_2 - \bar{z}a_2^\dagger a_1} \quad \text{for } z \in \mathbf{C}. \quad (31)$$

For the details of $U_J(z)$ see [18] or [12], [13].

Here we define unitary phase operators

$$V_j(\theta) = e^{i\theta N_j} \quad \text{for } \theta \in \mathbf{R}. \quad (32)$$

We can add a phase to coherent states like $|\alpha\rangle \longrightarrow |e^{i\theta}\alpha\rangle$ by making use of (32).

Let us make some mathematical preliminaries for the latter. We have easily

$$U_J(t)a_1U_J(t)^{-1} = \cos(|t|)a_1 - \frac{t\sin(|t|)}{|t|}a_2, \quad (33)$$

$$U_J(t)a_2U_J(t)^{-1} = \cos(|t|)a_2 + \frac{\bar{t}\sin(|t|)}{|t|}a_1, \quad (34)$$

so the map $(a_1, a_2) \longrightarrow (U_J(t)a_1U_J(t)^{-1}, U_J(t)a_2U_J(t)^{-1})$ is

$$(U_J(t)a_1U_J(t)^{-1}, U_J(t)a_2U_J(t)^{-1}) = (a_1, a_2) \begin{pmatrix} \cos(|t|) & \frac{\bar{t}\sin(|t|)}{|t|} \\ -\frac{t\sin(|t|)}{|t|} & \cos(|t|) \end{pmatrix} (\in SU(2)).$$

The exchange operator $S : \mathcal{H} \otimes \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}$ is given by

$$S = \sum_{a,b=0}^{\infty} (|a\rangle \otimes |b\rangle)(\langle b| \otimes \langle a|) = \sum_{a,b=0}^{\infty} |a\rangle\langle b| \otimes |b\rangle\langle a|. \quad (35)$$

However this is nothing but a mathematical formula.

Problem How can we construct this operator with some elementary ones ?

Here we meet some trouble. Let us explain. Even in this case we can define a shift operator

$$\Sigma : \mathcal{H} \longrightarrow \mathcal{H}, \quad \Sigma(|a\rangle) = |a+1\rangle \quad \text{for } a \geq 0.$$

But this operator is not unitary ! Namely, we cannot define a controlled shift operator like C_Σ .

Problem What is an operator corresponding to the controlled shift gate in the finite-dimensional cases ?

From now on let us change our purpose. We consider not all quantum states but small quantum states like coherent ones or squeezed-coherent ones.

The coherent states in (23) are exchanged by "elementary gates" in Quantum Optics

$$|z_1\rangle \otimes |z_2\rangle \longrightarrow |z_2\rangle \otimes |z_1\rangle. \quad (36)$$

The outline of the proof is as follows. By making use of $U_J(t)$ and (33), (34) we can show

$$|z_1\rangle \otimes |z_2\rangle \longrightarrow |\cos(|t|)z_1 + e^{i\theta}\sin(|t|)z_2\rangle \otimes |\cos(|t|)z_2 - e^{-i\theta}\sin(|t|)z_1\rangle$$

where $t = |t|e^{i\theta}$. By setting $|t| = \pi/2$ and removing the phases by making use of V_j ($j = 1, 2$) we obtain (36). For the details see [6].

Moreover we can perform what we have called "imperfect clone" for the coherent states

$$|z\rangle \otimes |0\rangle \longrightarrow |\cos(|t|)z\rangle \otimes |\sin(|t|)z\rangle. \quad (37)$$

These are a satisfactory result.

Next let us define a squeezed-coherent state

$$|(w, z)\rangle = S(w)D(z)|0\rangle \quad \text{for } w, z \in \mathbf{C}. \quad (38)$$

This state plays very important role in Holonomic Quantum Computation. However we cannot (at this stage) perform both exchange and "imperfect clone" for these states. This is dissatisfied to us.

Problem Find an (unitary) operator which exchange the squeezed-coherent states.

$$|(w_1, z_1)\rangle \otimes |(w_2, z_2)\rangle \longrightarrow |(w_2, z_2)\rangle \otimes |(w_1, z_1)\rangle.$$

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